Riesz transforms on compact quantum groups and strong solidity

Martijn Caspers – TU Delft
December 7, 2020

1. Classical Riesz transforms

Riesz transforms

Let $\mathcal{F}_2 : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be the (unitary) Fourier transform

$$\mathcal{F}_2(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{i\langle x, \xi \rangle} dx.$$

For $m \in L_\infty(\mathbb{R}^d)$ we define the Fourier multiplier

$$m(\nabla) := \mathcal{F}_2^{-1} \circ m \circ \mathcal{F}_2 : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d).$$

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$$m(\nabla) := \mathcal{F}_2^{-1} \circ m \circ \mathcal{F}_2 : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d).$$

Consider the function

$$R_j(\xi) = \frac{\xi_j}{\|\xi\|_2}, \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$

Then $R_j(\nabla)$ is called the Riesz transform!

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Side remark: Important when $m(\nabla)$ extends boundedly to $L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$
$\Rightarrow$ harmonic analysis, Calderón-Zygmund theory, ...

Riesz transforms

Consider the (positive) Laplace operator

$$\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

and the j-th coordinate gradient

$$\nabla_j = i \frac{\partial}{\partial x_j}.$$

We have

$$\nabla_j = \mathcal{F}_2^{-1} \circ \xi_j \circ \mathcal{F}_2, \qquad \Delta = \mathcal{F}_2^{-1} \circ \left(\sum_{j=1}^{d} \xi_j^2 \right) \circ \mathcal{F}_2 = \mathcal{F}_2^{-1} \circ \|\xi\|_2^2 \circ \mathcal{F}_2.$$

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Therefore we have for the Riesz transform:

$$\nabla_j \circ \Delta^{-\frac{1}{2}} = R_j(\nabla).$$

Riesz transforms

A Riesz transform

$$\nabla_j \circ \Delta^{-\frac{1}{2}}$$

can be defined for any Markov semi-group with generator Δ (details will come)!

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Non-exhaustive list of background:

- L_p-boundedness of $R_j(\nabla)$ (Calderón-Zygmund theory, Fefferman-Stein, ...).
- L_p-boundedness of Riesz transforms for Ornstein-Uhlenbeck semi-group (P.A. Meyer, Gundy, Pisier) and other semi-groups (Bakry).
- Links curvature of manifolds (Bakry, Li).
- L_p-boundedness of Riesz transforms for non-commutative Ornstein-Uhlenbeck semi-groups (Lust-Piquard).
- L_p-boundedness of types of non-commutative Riesz transforms (Junge, Mei, Parcet).
- ...

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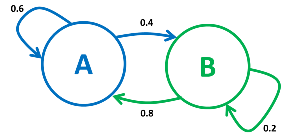
In this talk we shall not focus on L_p-boundedness but rather on applications to rigidity of von Neumann algebras!

2. Non-commutative Riesz transforms

Quantum Markov semi-groups

Markov process: Probabilistic process in which the subsequent state solely depends on the current state, and does not remember anything from the past.



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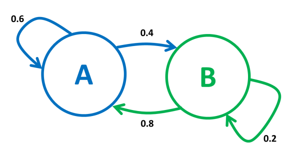

Quantum probability:

- Probability space $\Rightarrow$ von Neumann algebra with a trace.
- State $\Rightarrow$ density operators (positive, trace 1).
- Markov maps $\Rightarrow$ trace preserving normal unital completely positive (ucp) maps (quantum channels).

Quantum Markov semi-groups

Setup:

- $\mathcal{M}$ = von Neumann algebra, $A \subseteq \mathcal{M}$ a nice dense $*$-subalgebra.
- τ = normal faithful tracial state on $\mathcal{M}$
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A quantum Markov semi-group $(\Phi_t)_{t \geq 0}$ is a semi-group of normal unital completely positive (ucp) maps on a von Neumann algebra $\mathcal{M}$ that is point-strongly continuous. They interpolate to L_2-maps

$$\Phi_t^{(2)} : x\Omega_\tau \mapsto \Phi_t(x)\Omega_\tau.$$

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Unbounded generator $\Delta :\subseteq L_2(\mathcal{M}) \to L_2(\mathcal{M})$ such that,

$$\Phi_t^{(2)}(x) = \exp(-t\Delta).$$

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The Riesz transform is then a map

$$\nabla \Delta^{-\frac{1}{2}},$$

where the gradient ∇ shall be defined on the next slide.

Gradients of quantum Markov semi-group

Theorem (Cipriani-Sauvageot): there is a derivation ∇ that is the square root of Δ

Δ generator of a quantum Markov semi-group on $\mathcal{M}$ with some extra technical conditions omitted. There exists

- A subspace $\mathrm{Dom}(\nabla) \subseteq L_2(\mathcal{M})$ that is moreover a $*$-algebra,
- An $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{H}_\nabla$,
- A closable derivation $\nabla : \mathrm{Dom}(\nabla) \to \mathcal{H}_\nabla$,

such that $\nabla^* \overline{\nabla} = \Delta$.

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Definition gradient bimodule $\mathcal{H}_\nabla$. Assume for simplicity $\mathrm{Dom}(\nabla)$ is a $*$-subalgebra in $\mathrm{Dom}(\Delta)$. Consider inner products on $\mathrm{Dom}(\nabla) \otimes \mathrm{Dom}(\nabla)$ by

$$\langle a \otimes b, c \otimes d \rangle = \langle \Gamma(a, c) b, d \rangle_\tau,$$

with

$$\Gamma(a, c) = \frac{1}{2}(c^* \Delta(a) + \Delta(c)^* a - \Delta(c^* a)).$$

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$\mathcal{H}_\nabla$ is the completion of $\mathrm{Dom}(\nabla) \otimes \mathrm{Dom}(\nabla)$ modulo its degenerate part. Set,

$$x \cdot (a \otimes b) = xa \otimes b - x \otimes ab, \qquad (a \otimes b) \cdot x = a \otimes bx,$$
$$\nabla : \mathrm{Dom}(\nabla) \mapsto \mathcal{H}_\nabla : x \mapsto x \otimes 1.$$

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Leibniz rule: $\nabla(xy) = x\nabla(y) + \nabla(x)y$. Root: $\nabla^* \overline{\nabla} = \Delta$.

Non-commutative Riesz transforms

We now define

$$\nabla\Delta^{-\frac{1}{2}} : L_2(\mathcal{M}) \to \mathcal{H}_\nabla$$

called the Riesz transform.

The Riesz transform is isometric:

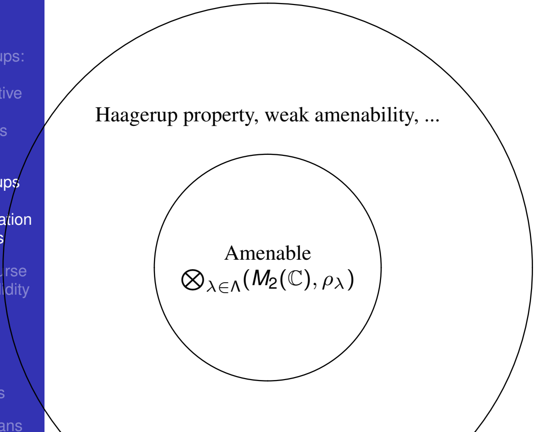
$$\begin{aligned}\langle \nabla\Delta^{-\frac{1}{2}}(x), \nabla\Delta^{-\frac{1}{2}}(x)\rangle_{H_\nabla} =& \langle \nabla^*\nabla\Delta^{-\frac{1}{2}}(x), \Delta^{-\frac{1}{2}}(x)\rangle_{L_2(\mathcal{M})} \\ =& \langle \Delta\Delta^{-\frac{1}{2}}(x), \Delta^{-\frac{1}{2}}(x)\rangle_{L_2(\mathcal{M})} \\ =& \langle x, x\rangle_{L_2(\mathcal{M})},\end{aligned}$$

(for x in a suitable dense domain).

3. Results: Markov semi-groups and approximation properties

On von Neumann algebras...

Property(T) $\Leftarrow$ Rigidity properties



$\Uparrow\Downarrow$ Quantum Markov semi-groups

$\Leftarrow$ Approximation properties

$\Uparrow\Downarrow$ Quantum Markov semi-groups

$\Leftarrow$ Amenability
(Connes: $\exists!$ amenable II_1-factor)

Quantum Markov semi-groups

Definition: A finite von Neumann algebra $\mathcal{M}$ has Haagerup property if there is a net of normal trace preserving ucp maps $\Phi_i : \mathcal{M} \rightarrow \mathcal{M}$ such that $\Phi_i^{(2)}$ is compact and $\forall \xi \in L_2(\mathcal{M})$ we have $\|\Phi_i^{(2)} \xi - \xi\| \rightarrow 0$.

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Theorem (C-Skalski '15, Jolissaint–Martin '04)

$\mathcal{M}$ has Haagerup property iff $\exists$ a quantum Markov semi-group with generator Δ with complete set of eigenvalues $\Delta_k, k \in \mathbb{N}$ (multiplicity allowed) such that

$$\Delta_k \to \infty.$$

Quantum Markov semi-groups

Definition: A von Neumann algebra $\mathcal{M}$ is amenable if there is a net of normal finite rank ucp maps $\Phi_i : \mathcal{M} \to \mathcal{M}$ such $\forall x \in \mathcal{M}$ we have $\Phi_i(x) \to x$ strongly.

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Theorem (Cipriani-Sauvageot '17, see also C 20)

$\mathcal{M}$ is amenable iff $\exists$ a quantum Markov semi-group with generator Δ with complete set of eigenvalues $\Delta_k, k \in \mathbb{N}$ (multiplicity allowed) such that

$$\Delta_k \gg \log(k).$$

4. Crash course strong solidity

Crash course strong solidity

$\mathcal{M}$ is called strong solidity [Ozawa-Popa '07] if for any diffuse amenable von Neumann subalgebra $\mathcal{B} \subseteq \mathcal{M}$ the normalizing algebra

$$\{u \in \mathcal{M} \text{ unitary } \mid u\mathcal{B}u^* = \mathcal{B}\}''$$

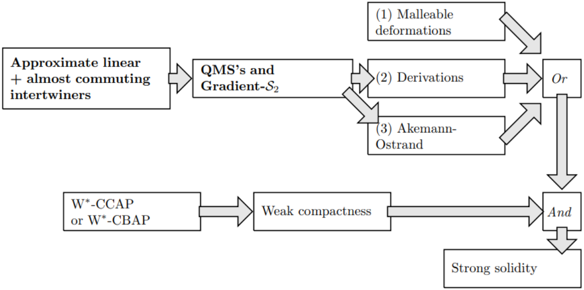
is again amenable.

Remark: In particular, strong solidity + non-amenability implies:

$$\mathcal{M} \not\simeq L_\infty(X) \rtimes \Lambda$$

or even absence of Cartan subalgebras.

Crash course strong solidity



Crash course Deformation-Rigidity theory

Definition: Akemann-Ostrand

A (finite) von Neumann algebra $\mathcal{M}$ has the Akemann-Ostrand property if there exists a dense unital C*-subalgebra $A \subseteq \mathcal{M}$ such that

1. A is locally reflexive.
2. There exists a ucp map

$$\theta : A \otimes_{\min} A^{\mathrm{op}} \to B(L_2(\mathcal{M}))$$

such that $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}$ is compact for all $a, b \in A$.

Suppose that $A \subseteq \mathcal{M}$ is a locally reflexive C*-subalgebra.

Proposition (C, Isono, Wasilewski)

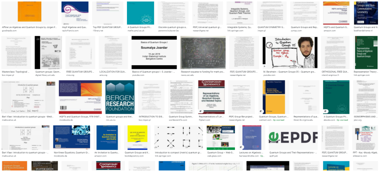
Suppose that

- H_∇ is weakly contained in $L_2(\mathcal{M}) \otimes L_2(\mathcal{M})$.
- $\Pi(a \otimes b^{\mathrm{op}}) \circ \nabla\Delta^{-\frac{1}{2}} = \nabla\Delta^{-\frac{1}{2}} \circ ab^{\mathrm{op}}$ is compact $\forall a, b \in A$.

Then $\mathcal{M}$ satisfies the Akemann-Ostrand property.

Proof. $\theta(a \otimes b^{\mathrm{op}}) := (\nabla\Delta^{-\frac{1}{2}})^* \Pi(a \otimes b^{\mathrm{op}}) \nabla\Delta^{-\frac{1}{2}}$ will do.

5. Quantum groups



Quantum groups

A compact quantum group is a pair $\mathbb{G} := (A, \Delta_A)$ with A a unital C*-algebra and $\Delta_A : A \to A \otimes A$ a comultiplication such that

$$(\Delta_A \otimes \mathrm{id})\Delta_A = (\mathrm{id} \otimes \Delta_A)\Delta_A,$$

and such that

$$\Delta_A(A)(A \otimes 1) \quad \text{and} \quad \Delta_A(A)(1 \otimes A),$$

are dense in $A \otimes A$.

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Free orthogonal quantum group O_N^+ is generated by a matrix $u = (u_{ij})_{ij}$ with the relations that u is unitary and $\overline{u} = u$. Compuplification:

$$\Delta_{O_N^+}(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Quantum groups

To a compact quantum group $\mathbb{G} = (A, \Delta_A)$ we associate:

Haar state $\varphi : A \to \mathbb{C}$ characterized by $(\varphi \otimes \mathrm{id}) \circ \Delta(x) = \varphi(x)1 = (\mathrm{id} \otimes \varphi) \circ \Delta(x)$.

Von Neumann algebra $L_\infty(\mathbb{G}) := \pi_\varphi(A)''$.

Corepresentation $\alpha \in A \otimes M_n(\mathbb{C})$ such that $(\Delta \otimes \mathrm{id})(\alpha) = \alpha_{13}\alpha_{23}$.

Irreducible if $\alpha(1 \otimes T) = (1 \otimes T)\alpha$ implies $T = 1$ for all $T \in M_n(\mathbb{C})$.

Tensor products and sums $\alpha \oplus \beta \in A \otimes (M_{n_\alpha}(\mathbb{C}) \oplus M_{n_\beta}(\mathbb{C}))$. $\alpha \otimes \beta = \alpha_{12}\beta_{13}$

Fusion rules $\alpha \otimes \beta \simeq \bigoplus_{\gamma \in \mathrm{Irr}(\mathbb{G})} n_\gamma \cdot \gamma$.

Matrix coefficients $\mathrm{Pol}(\mathbb{G}) := \{(\mathrm{id} \otimes \omega)(\alpha) \mid \omega \in M_{n_\alpha}(\mathbb{C})^*, \alpha \text{ corep}\}$.

Free orthogonal quantum groups

We know for free orthogonal quantum groups O_N^+:

- Non-amenability [Banica].
- Factoriality [Vaes–Vergnioux '03, see also Vaes (Appendix CFY) '12].
- Baum-Connes conjecture [Voigt '07].
- Connes embedding problem [Brannan, Collins, Vergnioux '15].
- Haagerup property and completely contractive approximation property [Brannan '11, Freslon '12, de Commer–Freslon–Yamashita '13].
- $L_\infty(O_N^+) \not\simeq L_\infty(\widehat{\mathbb{F}}_n)$ distinction from free factors [Brannan-Vergnioux, '16, see also Elzinga '20].
- Strong solidity [Fima-Vergnioux '14, Isono'13, C '20].

Quantum groups and QMS's

Quantum Markov semi-group of central multipliers is a Quantum Markov Semi-group $\Phi := (\Phi_t : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G}))_{t \geq 0}$ such that for every irreducible corepresentation α there exists $\Delta_\alpha \in \mathbb{R}_{\geq 0}$ with

$$(\Phi_t \otimes \mathrm{id})(\alpha) = \Delta_\alpha \alpha.$$

Φ has subexponential growth if for every $\alpha, \gamma \in \mathrm{Irr}(\mathbb{G})$,

$$\lim_{\beta \to \infty} \Delta_\beta = \infty, \qquad \lim_{\beta \to \infty} \sup_{\substack{\beta' \subseteq \alpha \otimes \beta \otimes \gamma, \\ \beta' \in \mathrm{Irr}(\mathbb{G})}} \left| \frac{\Delta_{\beta'}}{\Delta_\alpha} - 1 \right| = 0,$$

Theorem C, C-Isono-Wasilewski

If Φ has subexponential growth then $\Pi(a \otimes b^{\mathrm{op}}) \circ \nabla \Delta^{-\frac{1}{2}} = \nabla \Delta^{-\frac{1}{2}} \circ ab^{\mathrm{op}}$ is compact $\forall a, b \in \mathrm{Pol}(\mathbb{G})$.

Example: free orthogonal quantum groups

Representation theory of O_N^+ is given by $\mathbb{N}$ with fusion rules

$$\alpha \otimes \beta = |\alpha - \beta| \oplus |\alpha - \beta + 2| \oplus |\alpha - \beta + 4| \oplus \ldots \oplus \alpha + \beta.$$

[Banica].

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[Banica].

Theorem: There exists a quantum Markov semi-group $\Phi := (\Phi_t)_{t \geq 0}$ on $L_\infty(O_N^+)$ of central multipliers determined by

$$(\Phi_t \otimes \mathrm{id})(\alpha) = \exp\left(-t\Delta_\alpha\right)\alpha,$$

with

$$\Delta_\alpha = \frac{c'_\alpha(1)}{c_\alpha(1)} = \frac{\alpha}{\sqrt{N_q^2 - 4}} \frac{1 + q^{-2\alpha-2}}{1 - q^{-2\alpha-2}} + \frac{2}{(1 - q^2)\sqrt{N_q^2 - 4}},$$

and c_α the α-the Chebyshev polynomial. Moreover Φ is approximately linear.

[C.f. Brannan '11, Cipriani-Kula-Franz '15, Fima-Vergnioux '14, C-Skalski '15, Jolissaint-Martin '04].

Gradient-$\mathcal{S}_2 \Rightarrow$ Weak containment H_∇

Question: When is H_∇ weakly contained in $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$?

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Definition: $(\Phi_t)_{t\geq 0}$ is immediately gradient-$\mathcal{S}_2$ if for every $a, b \in \mathrm{Pol}(\mathbb{G})$ and $t \geq 0$ the mapping

$$x\Omega_\tau \mapsto \Phi_t(a\Delta(x)b + \Delta(axb) - a\Delta(xb) - \Delta(ax)b)\Omega_\tau,$$

is bounded $L_2(\mathbb{G}) \to L_2(\mathbb{G})$ and for $t > 0$ this mapping is moreover Hilbert-Schmidt.

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Theorem (C)

If Φ is immediately gradient-$\mathcal{S}_2$ then H_∇ is contained in a direct sum of copies of $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$. In particular H_∇ is weakly contained in $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$.

Property $\Rightarrow$ Gradient-$\mathcal{S}_2$ $\Rightarrow$ Weak containment H_∇

For $\alpha, \beta, \gamma \in \mathrm{Irr}(\mathbb{G})$, $\beta_2 \subseteq \alpha \otimes \beta \otimes \gamma$ we define

$$
\begin{aligned}
L_\beta^{\alpha,\gamma} &= \{(\beta_1, \beta_2) \in \mathrm{Irr}(\mathbb{G}) \times \mathrm{Irr}(\mathbb{G}) \mid \beta_1 \subseteq \alpha \otimes \beta, \beta_2 \subseteq \beta_1 \otimes \gamma\},\\
R_\beta^{\alpha,\gamma} &= \{(\beta_1, \beta_2) \in \mathrm{Irr}(\mathbb{G}) \times \mathrm{Irr}(\mathbb{G}) \mid \beta_1 \subseteq \beta \otimes \gamma, \beta_2 \subseteq \alpha \otimes \beta_1\},\\
L_{\beta,\beta_2}^{\alpha,\gamma} &= \{\beta_1 \in \mathrm{Irr}(\mathbb{G}) \mid (\beta_1, \beta_2) \in L_\beta^{\alpha,\gamma}\},\\
R_{\beta,\beta_2}^{\alpha,\gamma} &= \{\beta_1 \in \mathrm{Irr}(\mathbb{G}) \mid (\beta_1, \beta_2) \in R_\beta^{\alpha,\gamma}\}.
\end{aligned}
$$

Definition 2.2. We say that Φ is *approximately linear with almost commuting intertwiners* if the following holds. For every $\alpha, \gamma \in \mathrm{Irr}(\mathbb{G})$ there exists a finite set $A_{00} := A_{00}(\alpha, \gamma) \subseteq \mathrm{Irr}(\mathbb{G})$ such that for every $\beta \in \mathrm{Irr}(\mathbb{G}) \backslash A_{00}$ and $\beta_2 \subseteq \alpha \otimes \beta \otimes \gamma$ there exist bijections (called the *v-maps*),

$$v^{\alpha,\gamma}(\,\cdot\,; \beta, \beta_2) := v(\,\cdot\,; \beta, \beta_2) : L_{\beta,\beta_2}^{\alpha,\gamma} \to R_{\beta,\beta_2}^{\alpha,\gamma},$$

such that the following holds. There exists a set $A \subseteq \mathrm{Irr}(\mathbb{G}) \backslash A_{00}$ and a constant $C := C(\alpha, \gamma) > 0$ such that

(1) For all $\beta \in A$, $(\beta_1, \beta_2) \in L_\beta^{\alpha,\gamma}$ we have

$$|\Delta_\beta - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta,\beta_2)} + \Delta_{\beta_2}| \leq C \,\mathrm{qdim}(\beta)^{-1}, \tag{2.1}$$

and

$$|\Delta_\beta - \Delta_{\beta_1}| \leq C. \tag{2.2}$$

For all $\beta \in \mathrm{Irr}(\mathbb{G}) \backslash (A \cup A_{00})$, $(\beta_1, \beta_2) \in L_\beta^{\alpha,\gamma}$ we have

$$\Delta_\beta - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta,\beta_2)} + \Delta_{\beta_2} = 0. \tag{2.3}$$

(2) For all $\beta \in A$, $(\beta_1, \beta_2) \in L_\beta^{\alpha,\gamma}$ we have

$$\inf_{z \in \mathbb{T}} \| V_{\beta_2}^{\beta_1,\gamma}(V_{\beta_1}^{\alpha,\beta} \otimes \mathrm{id}_\gamma) - z V_{\beta_2}^{\alpha, v(\beta_1;\beta,\beta_2)}(\mathrm{id}_\alpha \otimes V_{v(\beta_1;\beta,\beta_2)}^{\beta,\gamma}) \| \leq C \,\mathrm{qdim}(\beta)^{-1}. \tag{2.4}$$

For all $\beta \in \mathrm{Irr}(\mathbb{G}) \backslash (A \cup A_{00})$, $(\beta_1, \beta_2) \in L_\beta^{\alpha,\gamma}$ we have

$$\inf_{z \in \mathbb{T}} \| V_{\beta_2}^{\beta_1,\gamma}(V_{\beta_1}^{\alpha,\beta} \otimes \mathrm{id}_\gamma) - z V_{\beta_2}^{\alpha, v(\beta_1;\beta,\beta_2)}(\mathrm{id}_\alpha \otimes V_{v(\beta_1;\beta,\beta_2)}^{\beta,\gamma}) \| = 0. \tag{2.5}$$

(3) There exists a polynomial P such that for every $N \in \mathbb{N}$ we have

$$\#\{\beta \in A \mid \Delta_\beta < N\} \leq P(N). \tag{2.6}$$

and we have that $\beta \mapsto \delta(\beta \in A)\, \mathrm{qdim}(\beta)^{-1}$ is square summable.

Property $\Rightarrow$ Gradient-$\mathcal{S}_2$ $\Rightarrow$ Weak containment H_∇

Theorem (C)

If Φ is appproximately linear with almost commuting intertwiners then it is immediately gradient-$\mathcal{S}_2$ and consequently H_∇ is weakly contained in $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$.

Theorem (C)

Φ on O_N^+ (or $SU_q(2)$) as before is approximately linear with almost commuting intertwiners.

Theorem (C)

Appproximately linearity with almost commuting intertwiners is stable under

- Monoidal equivalence.
- Taking dual quantum subgroups.
- Free products.
- Products with finite (dimensional quantum) groups.
- Free wreath products with S_N^+, $N \geq 5$ [Lemeux-Tarrago, see also Bichon, Tarrago-Wahl].

Conclusion

Theorem (C)

If Φ witnesses the central ACPAP of a compact quantum group of Kac type $\mathbb{G}$ and is approximately linear with almost commuting intertwiners then it is strongly solid.

Corollary: examples

Examples include anything that can be constructed from $SU_q(2)$ and taking

- Monoidal equivalences.
- Dual quantum subgroups.
- Free products.
- Products with finite (dimensional quantum) groups.
- Free wreath products with S_N^+, $N \geq 5$.

Includes all non-colored non-crossing partition quantum groups (=free orthogonal easy QGs).

Conclusion in non-Kac case

Theorem (C)

For any $F \in GL_N(\mathbb{C})$ the von Neumann algebras $L_\infty(O_N^+(F))$ and $L_\infty(U_N^+(F))$ are strongly solid.

Conclusion in non-Kac case

Theorem (C)

For any $F \in GL_N(\mathbb{C})$ the von Neumann algebras $L_\infty(O_N^+(F))$ and $L_\infty(U_N^+(F))$ are strongly solid.

Recall strong solidity: For any diffuse faithfully expected amenable von Neumann subalgebra $B \subseteq L_\infty(O_N^+)$ the normalizing algebra

$$\{u \in L_\infty(O_N^+) \mid uBu^* = B\}''$$

is again amenable.

6. q-Gaussians

Gaussian algebras

Fix $H = \mathbb{C}^n$ finite dimensional Hilbert space $\Rightarrow$ Set Fock space:

$$F = \mathbb{C}\Omega \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus (H \otimes H \otimes H \otimes H) \oplus \dots$$

Gaussian algebras

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$$F = \mathbb{C}\Omega \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus (H \otimes H \otimes H \otimes H) \oplus \ldots$$

Consider creation and annihilation operators:

$$\begin{aligned} a^*(\xi) : & \eta_1 \otimes \ldots \otimes \eta_n = \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n, \\ a(\xi) : & \eta_1 \otimes \ldots \otimes \eta_n = \quad \langle \xi, \eta_1 \rangle \eta_2 \otimes \ldots \otimes \eta_n. \end{aligned}$$

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Voiculescu's free Gaussian algebra: $\Gamma(H) = \{ a(\xi) + a^*(\xi) \mid \xi \in H \}''$.

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Remark: Can q-symmetrize the inner product $\Rightarrow$ q-Gaussian algebras $q \in [-1, 1]$ (Bozejko-Speicher, 1993).

- $q = 1$ bosonic.
- $q = -1$ fermionic, harmonic oscillator.
- $q = 0$ free, c.f. above.

Quantum Markov semi-groups

Example (continued): $\xi_1 \otimes \ldots \otimes \xi_n \in F$ in the Fock space. Then $\exists$ $W(\xi_1 \otimes \ldots \xi_n) \in \Gamma(H)$ such that

$$W(\xi_1 \otimes \ldots \otimes \xi_n)\Omega = \xi_1 \otimes \ldots \otimes \xi_n.$$

Note: this is quantization.

Quantum Markov semi-groups

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The Fock space semi-group

$$\Phi_t^{(2)} : F \to F : \xi_1 \otimes \ldots \xi_n \mapsto e^{-tn}\xi_1 \otimes \ldots \otimes \xi_n.$$

lifts to the algebra level

$$\Phi_t : \Gamma(H) \to \Gamma(H) : W(\xi_1 \otimes \ldots \otimes \xi_n) \mapsto e^{-tn}W(\xi_1 \otimes \ldots \otimes \xi_n).$$

Note: this is second quantization.

Quantum Markov semi-groups

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Note: this is second quantization.

$(\Phi_t)_{t \geq 0}$ is a quantum Markov semi-group (Ornstein-Uhlenbeck semi-group).

Quantum Markov semi-groups

Theorem (C-Isono-Wasilewski)

$(\Phi_t)_{t \geq 0}$ is immediately gradient-$\mathcal{S}_2$ if

$$|q| \leq \dim(H)^{-1/2},$$

and consequently $\Gamma_q(H)$ has the Akemann-Ostrand property.

Theorem (Shlyakhtenko)

$\Gamma_q(H)$ has the Akemann-Ostrand property for $|q| < \sqrt{2} - 1$ and H finite dimensional.

Theorem (Avsec)

$\Gamma_q(H)$ is strongly solid for all $q \in (-1, 1)$ and H finite dimensional.

Open questions:

- Strong solidity with H infinite dimensional.
- Akemann-Ostrand property beyond the range $|q| \leq \max(\sqrt{2} - 1, \dim(H)^{1/2})$.